Math 214 - Fall 2019 - Quiz 3

Answer any ONE of the following questions:

1. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Compute e^{At}.

   Hint: The eigenvectors of A are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2. Find an orthogonal set of vectors which spans the same subspace as: Span $(\{[1, 2, 3], [0, 1, 5]\})$.

3. Choose the value of x such that the vectors below form an orthogonal set. Then, express the vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal vectors:

$$\bar{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1 \\ 1 \\ x \end{bmatrix}.$$

Math 214 - Fall 2019 - Quiz 3 - Solutions

Answer any ONE of the following questions:

1. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Compute e^{At}.

   Hint: The eigenvectors of A are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   The following computations are straight forward:

   (a) Since the matrix is triangular, the eigenvalues are the diagonal elements:
   $\boldsymbol{\lambda_1 = 2}, \quad \boldsymbol{\lambda_2 = 1}$

   (b) The corresponding eigenvectors are: $\bar{\mathbf{v}}_{[\lambda=\mathbf{2}]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{v}}_{[\lambda=\mathbf{1}]} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (c) The relevant matrices, then, are:

   $$\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{S^{-1}} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{e^{Dt}} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

   (d) $$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
   $$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & e^{2t} \\ 0 & -e^{t} \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{2t} - e^{t} \\ 0 & e^{t} \end{bmatrix}$$

2. Find an orthogonal set of vectors which spans the same subspace as: Span $(\{[1,2,3],[0,1,5]\})$.

   Row Reducing: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2+c & 3+5c \\ 0 & 1 & 5 \end{bmatrix}$.

   The two vectors $[1, 2+c, 3+5c]$ and $[0,1,5]$ are orthogonal if their dot product is equal to 0. So:

   $$\mathbf{(2+c) + 5(3+5c) = 26c + 17 = 0} \quad \Rightarrow \quad \mathbf{c = -\tfrac{17}{26}}$$

   So our orthogonal vectors are:

   $[\mathbf{1, 2+c, 3+5c}]$, $[\mathbf{0, 1, 5}]$, or $[\mathbf{1}, \tfrac{\mathbf{35}}{\mathbf{26}}, -\tfrac{\mathbf{7}}{\mathbf{26}}]$, $[\mathbf{0,1,5}]$.

   Since the allowable elementary row operations do not change the span of the vectors, we conclude that:

   $$\textbf{Span } (\{[1,2,3],[0,1,5]\}) = \textbf{Span } \left(\left\{[1, \tfrac{35}{26}, -\tfrac{7}{26}], [0,1,5]\right\}\right)$$

3. Choose the value of x such that the vectors below form an orthogonal set. Then, express the vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal vectors:

$$\bar{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{v}_3 = \begin{bmatrix} -1 \\ 1 \\ x \end{bmatrix}.$$

First, note that $\bar{v}_1 \cdot \bar{v}_2 = 0$ and $\bar{v}_2 \cdot \bar{v}_3 = 0$ for ALL values of x. Further, $\bar{v}_1 \cdot \bar{v}_3 = x - 4$, so $\bar{v}_1 \cdot \bar{v}_3 = 0$ if and only if $x = 4$. Therefore, **the three vectors only form an orthogonal set when $\mathbf{x = 4}$**.

Next, using the equations $c_k = \frac{\bar{w} \cdot \bar{v}_k}{||\bar{v}_k||^2}$, (and assuming that $x = 4$), where:

$$\mathbf{||\bar{v}_1||^2 = 9}, \qquad \mathbf{||\bar{v}_2||^2 = 2}, \qquad \mathbf{||\bar{v}_3||^2 = 18}, \quad \text{we have:}$$

$$c_1 = \frac{\bar{w} \cdot \bar{v}_1}{||\bar{v}_1||^2} = \frac{1}{9} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \mathbf{\frac{10}{9}},$$

$$c_2 = \frac{\bar{w} \cdot \bar{v}_2}{||\bar{v}_2||^2} = \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{\frac{1}{2}},$$

$$c_3 = \frac{\bar{w} \cdot \bar{v}_3}{||\bar{v}_3||^2} = \frac{1}{18} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{\frac{13}{18}}.$$

We can easily verify that:

$$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \mathbf{\frac{10}{9}} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \mathbf{\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{\frac{13}{18}} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = \mathbf{c_1 \bar{v}_1 + c_2 \bar{v}_2 + c_3 \bar{v}_3}$$