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Math 214 - Fall 2019 - Midterm Exam 2 - Solutions

1. For each statement below, CLEARLY write **T** for True, **F** for False:

   (a) **FALSE** : Let $\bar{v}$ be an eigenvector for A. Then $\bar{v}$ is an eigenvector for A^T (A-Transpose).

   (b) **FALSE** : For a 2 by 4 matrix A, Nullity$(A) \leq$ Rank(A).

   (c) **TRUE** : Let $A \sim B$, for ANY similarity transformation. Then the eigenvalues for A are the same as the eigenvalues for B.

   (d) **FALSE** : The determinant of a 4 by 4 matrix, A, consists of the sum of 8 terms, where each term is a product of 4 entries from A.

   (e) **FALSE** : Let $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x+1 \\ y \\ x-2y \end{bmatrix}$. Then T is a linear transformation from R^2 to R^3.

   (f) **FALSE** : $\begin{bmatrix} 3 & -5 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 5 \\ -2 & 3 \end{bmatrix}$.

   (g) **FALSE** : For any n by n matrix, A, $Row(A) + Col(A) = n$.

   (h) **TRUE** : For any 2 by 2 matrix, A, the characteristic polynomial is given by: $\lambda^2 - Tr(A)\lambda + Det(A)$.

   (i) **TRUE** : Let A and B be n by n matricies. Then $Det(AB) = Det(A)Det(B)$ and $Tr(A+B) = Tr(A) + Tr(B)$.

   (j) **FALSE** : Let A be an n by n matrix for which $A = A^{-1} = A^T$. Then A must be the identity, I_n.

2. Determinants:

(a) Compute the determinant of $A = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 5 & -1 \end{bmatrix}$

$$Det(A) = Det\left(\begin{bmatrix} \mathbf{3} & \mathbf{-2} & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 5 & -1 \end{bmatrix}\right)$$

$$= \mathbf{3} \cdot (-\mathbf{1})^{(\mathbf{1+1})}\mathbf{Det}\left(\begin{bmatrix} 0 & 3 & 0 \\ \mathbf{1} & 0 & 3 \\ 0 & 5 & -1 \end{bmatrix}\right) - \mathbf{2} \cdot (-\mathbf{1})^{(\mathbf{1+2})}\mathbf{Det}\left(\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & \mathbf{3} \\ 2 & 5 & -1 \end{bmatrix}\right)$$

$$= \mathbf{3} \cdot (-\mathbf{1})^{(\mathbf{1+1})} \cdot \mathbf{1} \cdot (-\mathbf{1})^{(\mathbf{1+2})}\mathbf{Det}\left(\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}\right) - \mathbf{2} \cdot (-\mathbf{1})^{(\mathbf{1+2})} \cdot \mathbf{3} \cdot (-\mathbf{1})^{(\mathbf{2+3})}\mathbf{Det}\left(\begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}\right)$$

$$= \mathbf{3} \cdot (-\mathbf{1})^{(\mathbf{1+1})} \cdot \mathbf{1} \cdot (-\mathbf{1})^{(\mathbf{1+2})} \cdot -\mathbf{3} - \mathbf{2} \cdot (-\mathbf{1})^{(\mathbf{1+2})} \cdot \mathbf{3} \cdot (-\mathbf{1})^{(\mathbf{2+3})} \cdot \mathbf{4}$$

$$= \mathbf{9} - \mathbf{24} = -\mathbf{15}$$

(b) A different matrix, B, of the same size as A, has determinant $det(B) = \frac{1}{2}$. Compute the values:

$$det(B^T A^{-2}) = ... = det(B^T) * \frac{1}{det(A)^2} = det(B) * \frac{1}{det(A)^2} = \frac{1}{2} \cdot \frac{1}{225} = \mathbf{\frac{1}{450}}$$

$$det(A^{-1}B^{-1}AB) = ... = \frac{1}{det(A)} \cdot \frac{1}{det(B)} \cdot det(A) \cdot det(A) = \mathbf{1}$$

(c) Prove: An orthogonal matrix must have determinant equal to either 1 or -1.

By definition, an orthogonal matrix is one for which the inverse is equal to the transpose. That is, A is orthogonal if and only if $A^{-1} = A^T$. We can then compute:

$$det(A) \cdot det(A) = det(A) \cdot det(A^T) = det(A) \cdot det(A^{-1}) = det(A) \cdot \frac{1}{det(A)} = 1$$

Since $det(A)^2 = 1$, $\mathbf{det(A) = \pm 1}$.

3. Eigen-things!

(a) $A = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 5 & 0 \\ 10 & 0 & -1 \end{bmatrix}$. One of the eigenvectors of A is: $\bar{v} = \begin{bmatrix} 7 \\ 7 \\ 10 \end{bmatrix}$

Find all eigenvalues and eigenvectors. Hint: Clever thought, and a few theorems from class, will minimize the amount of work needed!

First: $A\bar{v} = \begin{bmatrix} 42 \\ 42 \\ 60 \end{bmatrix} = 6\bar{v}$, so: $\lambda_{\mathbf{1}} = \mathbf{6}, \bar{\mathbf{v}}_{\mathbf{1}} = \begin{bmatrix} 7 \\ 7 \\ 10 \end{bmatrix}$.

Second: Computing the characteristic equation: $Det(A-\lambda I) = (-1-\lambda)Det\left(\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}\right)$.

So $\lambda_{\mathbf{2}} = -\mathbf{1}$. Since $\lambda_1 + \lambda_2 + \lambda_3 = 6 - 1 + \lambda_3 = Tr(A) = 8$, we have $\lambda_{\mathbf{3}} = \mathbf{3}$

Third: Solving the homogenous system $(A - \lambda I)\bar{v} = \bar{0}$ for $\lambda_2 = -1$ and $\lambda_3 = 3$, we see:

$$\bar{\mathbf{v}}_{\mathbf{2}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \bar{\mathbf{v}}_{\mathbf{3}} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

(b) Suppose $B = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ has eigenvalue/eigenvector pairs: $\lambda = 2, \bar{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\lambda = -1, \bar{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Use this information to compute the value B^5.

$$S = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B^5 = SD^5S^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 32 & 64 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 66 \\ 0 & -1 \end{bmatrix}$$

(c) Give an example of a 5 by 5 matrix whose eigenvalues are $3, 1, 4, 1, 5$, and which has exactly 11 non-zero elements.

Any triangular matrix has it's eigenvalues along the principle diagonal. So, for instance:

$$\begin{bmatrix} 3 & 6 & 0 & 10 & 0 \\ 0 & 1 & 7 & 0 & 11 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

4. Subspaces

$A = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 5 & -1 \end{bmatrix}$. Row reducing A gives: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. Determine:

(a) The Row Space of A, $Row(A) = ... = \textbf{span}(\{[\mathbf{1\ 0\ 0\ 2}], [\mathbf{0\ 1\ 0\ 3}], [\mathbf{0\ 0\ 1\ -1}]\})$

(b) The Column Space of A, $Col(A) =$

Since, in the row reduced form, column 4 is a linear combination of the other columns, it may be deleted. So we can also delete column 4 in the original matrix. This leaves the linearly independent set below as the basis of $Col(A)$:

$$\mathbf{Col(A)} = \textbf{span}\left(\left\{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}\right\}\right)$$

(c) The Null Space of A, $Nul(A) =$

The solution for the augmented matrix: $[A|0] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$

This has solution vector: $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -2v_4 \\ -3v_4 \\ v_4 \\ v_4 \end{bmatrix} = v_4 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 1 \end{bmatrix}$

So: $\mathbf{Nul(A)} = \textbf{span}\left(\left\{\begin{matrix} -2 \\ -3 \\ 1 \\ 1 \end{matrix}\right\}\right)$

(d) State and verify the Rank Theorem for this particular matrix:

For a matrix $A_{m,n}$, $rank(A) + nullity(A) = n$. Since rank(A) (the dimension of the row space or the column space) is equal to 3, and the nullity (the dimension of the null space) is equal to 1, we have: $\mathbf{rank(A) + nullity(A) = 3 + 1 = 4 = n}$

(e) Give an example of a matrix, A, for which $Row(A) = Col(A)$:

The simplest that comes to mind is: $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. Inverses: $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & x & 0 \\ -4 & 0 & 2 \end{bmatrix}$

(a) For which values of x does an inverse, A^{-1}, exist?

$Det(A) = 4x - 16x = -12x$. So $Det(A) = 0$ if and only if $x = 0$. So for all $\mathbf{x \neq 0}$, A^{-1}, exists.

(b) Compute the matrix A^{-1}, assuming a permissible value for x:

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 0 & -4 & 1 & 0 & 0 \\ 0 & x & 0 & 0 & 1 & 0 \\ -4 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & x & 0 & 0 & 1 & 0 \\ -4 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & x & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 2 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{x} & 0 \\ 0 & 0 & -6 & 2 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{x} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{6} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{x} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{6} \end{array}\right] \Rightarrow$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & \frac{1}{x} & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{6} \end{bmatrix} = -\mathbf{\frac{1}{6}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -\frac{6}{x} & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

(c) Prove: For an invertible n by n matrix, B, if v is an eigenvector of B, then v is also an eigenvector of B^{-1}.
Let v be an eigenvector of B with corresponding eigenvalue λ. Then:

$\mathbf{Bv} = \lambda\mathbf{v}$.

Multiplying both sides on the left by B^{-1}, we have: $\mathbf{v} = \mathbf{B^{-1}Bv} = \mathbf{B^{-1}}\lambda\mathbf{v}$.

Dividing both sides by λ, we have:
$\mathbf{\frac{1}{\lambda}Iv} = \mathbf{\frac{1}{\lambda}B^{-1}Bv} \quad \Rightarrow \quad \mathbf{\frac{1}{\lambda}v} = \mathbf{B^{-1}v}$.

Since $\mathbf{B^{-1}v} = \mathbf{\frac{1}{\lambda}v}$, we conclude that v is an eigenvector of B corresponding to eigenvalue $\frac{1}{\lambda}$.

NOTE: Did you forget to at least wonder if dividing by λ is always allowed?? What if $\lambda = 0$?? But wait: if $\lambda = 0$, then $det(B) = 0$, and so B^{-1} doesn't exist! Since our problem assumes that B^{-1} exists, we must automatically conclude that $\lambda \neq 0$, and so it IS ok to divide by it! Whew... close one!

6. Seemingly random questions...

(a) Compute the determinant matrix P, where:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

By first stating the determinants of each individual matrix.

If we identify the matrices above by: $P = A_1A_2A_3A_4A_5A_6A_7$ we see that $A_1 ... A_6$ are all elementary matrices corresponding to elementary row operations. A_1 and A_4 represent "add c Row i into Row j, so both have determinant 1. A_2 corresponds to "multiply row i by c" and has determinant equal to 2. A_3 and A_5 both represent "swap row i with row j", and so i) each is equal to it's own inverse, and ii) both have determinant equal to -1. Finally, A_6 is a triangular matrix with determinant equal to the product of elements on the main diagonal: $1 \cdot 3 \cdot 6 = 18$. We conclude:

$$\begin{aligned} Det(P) &= Det(A_1)Det(A_2)Det(A_3)Det(A_4)Det(A_5)Det(A_6) \\ &= 1 \cdot 2 \cdot -1 \cdot 1 \cdot -1 \cdot 18 = \mathbf{36}. \end{aligned}$$

(b) Given an invertible matrix, Q, and a matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. What are the eigenvalues, trace, and determinant of the matrix R given by:

$$R = Q(A^3 - 2A^2 + 7A - 3I)Q^{-1}$$

It's easy to verify that for A, we have: i) $\lambda_1 = 3, v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and ii) $\lambda_2 = -1, v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Defining $p(A) = A^3 - 2A^2 + 7A - 3I$, we easily compute the eigenvalues of $p(A)$ as i) $\mathbf{p(\lambda_1) = p(3) = 27}$, and ii) $\mathbf{p(\lambda_2) = p(-1) = -13}$. From this we compute the trace of $p(A)$ is $\mathbf{Tr(p(A)) = 27 - 13 = 14}$, and the determinant of $p(A)$ is $\mathbf{Det(p(A)) = 27 \cdot -13 = -351}$.

Since R is found as a similarity transformation of $p(A)$, and we know that similarity transformations preserve i) characteristic equations, eigenvalues, trace, and determinant, we see that for R:

$$\boldsymbol{\lambda_1 = 27}, \quad \boldsymbol{\lambda_2 = -13}, \quad \mathbf{Tr(R) = 14}, \quad \mathbf{Det(R) = -351}.$$

(c) Give an example of a 2 by 2 matrix which only has one eigenvector.

$\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ has eigenvalues $\lambda = 5, 5$. Computing the eigenector(s), we find:

$(A - \lambda I)v = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v = 0$ The only solution to this is:

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.