1	2	3	4	5	6	Total
/ 10	/ 12	/ 12	12	/ 12	/ 12	/ 70

Math 214 - Fall 2019 - Midterm Exam 1 - Solutions

Remember: No notes, No Calculators, No People!

1. Let $\bar{v} = [2, -1, 5]$ and $\bar{w} = [0, 7, -4]$. Compute the following, or explain why this is not possible:

   (a) $\bar{v} \cdot \bar{w}$:

   $\bar{v} \cdot \bar{w} = 0 - 7 - 20 = \mathbf{-27}$

   (b) $\bar{v} \times \bar{w}$:

   $$\bar{v} \times \bar{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 5 \\ 0 & 7 & -4 \end{vmatrix} = (4 - 35)\hat{i} + (0 - -8)\hat{j} + (14 - 0)\hat{k} = [\mathbf{-31}, \mathbf{8}, \mathbf{14}]$$

   (c) $\text{Proj}_{\bar{w}}\bar{v}$:

   $\text{Proj}_{\bar{w}}\bar{v} = \frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}}\bar{w} = \frac{\mathbf{-27}}{\mathbf{65}} [\mathbf{0}, \mathbf{7}, \mathbf{-4}]$

   (d) The unit vector in the direction of $\bar{w}$:

   $= \frac{\bar{w}}{||\bar{w}||} = \frac{[0,7,-4]}{(49+16)} = \frac{\mathbf{1}}{\mathbf{65}} [\mathbf{0}, \mathbf{7}, \mathbf{-4}]$

   (e) $||\bar{w} + \bar{v}||$:

   $||\bar{w} + \bar{v}|| = ||[0, 7, -4] + [2, -1, 5]|| = ||[2, 6, 1]|| = \sqrt{4 + 36 + 1} = \sqrt{\mathbf{41}}$

2. Solve each of the following linear systems by i) writing the appropriate Augmented Matrix, ii) writing the corresponding Reduced Row Echelon Form, iii) identifying any solution(s), iv) writing a brief statement verifying the Rank Theorem for the system.

(a) $x + 3y - z = 6, \qquad -2x + 5z = 0, \qquad 4x - 2y + 8z = 34$

$$\left[\begin{array}{rrr|r} 1 & 3 & -1 & 6 \\ -2 & 0 & 5 & 0 \\ 4 & -2 & 8 & 34 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|r} 1 & 3 & -1 & 6 \\ 0 & 6 & 3 & 12 \\ 0 & -14 & 12 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|r} 1 & 3 & -1 & 6 \\ 0 & 1 & 1/2 & 2 \\ 0 & -14 & 12 & 10 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -5/2 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 19 & 38 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -5/2 & 0 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution: $[\mathbf{x}, \mathbf{y}, \mathbf{z}] = [\mathbf{5}, \mathbf{1}, \mathbf{2}]$.

Rank Theorem: Rank + Number of Free Variables = Number of Variables:
$\mathbf{3 + 0 = 3}$ ✓

(b) $w + 3x + 3y - 16z = -14, \qquad -3w - 8x - 7y + 43z = 36$

$$\left[\begin{array}{rrrr|r} 1 & 3 & 3 & -16 & -14 \\ -3 & -8 & -7 & 43 & 36 \end{array}\right] \Rightarrow \left[\begin{array}{rrrr|r} 1 & 3 & 3 & -16 & -14 \\ 0 & 1 & 2 & -5 & -6 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & -3 & -1 & 4 \\ 0 & 1 & 2 & -5 & -6 \end{array}\right]$$

Solution: $$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y + z + 4 \\ -2y + 5z - 6 \\ y \\ z \end{bmatrix} = \mathbf{y} \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \mathbf{z} \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \\ 0 \\ 0 \end{bmatrix}$$

Rank Theorem: Rank + Number of Free Variables = Number of Variables:
$\mathbf{2 + 2 = 4}$ ✓

3. Determine the plane in R^3 which contains the points $P = (0, 1, 3)$, $Q = (-2, 0, 5)$, $R = (4, -3, 0)$. Express your answer in both i) general form and ii) vector form.

Define the following vectors which lie in this plane:
$\bar{v} = PQ = Q - P = [-2, -1, 2]$ $\quad\quad$ $\bar{w} = PR = R - P = [4, -4, -3]$. Then:

General Form: $\bar{n} = \bar{v} \times \bar{w} = [11, 2, 12]$; Then $\bar{n} \cdot (\bar{x} - P) = 0$

$[11, 2, 12] \cdot ([x, y, z] - [0, 1, 3]) = 11x + 2(y-1) + 12(z-3) = 0 \Rightarrow \mathbf{11x + 2y + 12z = 38}$

Vector Form: $\bar{x} = P + t\bar{v} + s\bar{w}$, or:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + \mathbf{t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + \mathbf{s} \begin{bmatrix} 4 \\ -4 \\ -3 \end{bmatrix}$$

Determine the line in R^5 which contains the points $P = (4, 0, 3, 0, 3)$ and $Q = (-1, -2, 0, 1, 2)$ in vector form.

$$\bar{x} = P + t(Q - P) \Rightarrow \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} + \mathbf{t} \begin{bmatrix} -5 \\ -2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$$

4. Prove each of the following statements:

(a) Assume that the three vectors $\bar{u}, \bar{v}, \bar{w}$ are linearly independent. Then the vectors $\bar{u}+\bar{v}$, $\bar{u}-\bar{w}$, $\bar{v}+2\bar{w}$ are linearly independent.

We construct the test equation:

$$\mathbf{c_1(\bar{u}+\bar{v})+c_2(\bar{u}-\bar{w})+c_3(\bar{v}+2\bar{w})=\bar{0}}.$$

Then: $\mathbf{(c_1+c_2)\bar{u}+(c_1+c_3)\bar{v}+(2c_3-c_2)\bar{w}=\bar{0}}$.

Since $\bar{u}, \bar{v}, \bar{w}$ are linearly independent, $\mathbf{c_1+c_2=c_1+c_3=2c_3-c_2=0}$.

The only solution is $\mathbf{c_1=c_2=c_3=0}$. Therefore:

$\mathbf{\{\bar{u}+\bar{v}, \bar{u}-\bar{w}, \bar{v}+2\bar{w}\}}$ **is a linearly independent set of vectors.**

(b) Assume that $V=\{\bar{v}_1, \bar{v}_2, ...\bar{v}_n\}$ is a linearly dependent set of vectors in R^n. If we add another vector, $\bar{v}_{n+1} \in R^n$, we have a new set: $W=\{\bar{v}_1, \bar{v}_2, ...\bar{v}_n, \bar{v}_{n+1}\}$. Prove that W is a linearly dependent set as well.

If set V is a set of linearly dependent vectors, then there exists a set of scalars, $c_1, c_2, ..., c_n$ which are not ALL equal to 0 for which:

$$\mathbf{c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3+...+c_n\bar{v}_n=\bar{0}}$$

Without loss of generality, (WLOG), let $\mathbf{c_1 \neq 0}$

If we add vector $\bar{v}_{n+1}$ with coefficient $c_{n+1}=0$, we now have:

$\mathbf{c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3+...+c_n\bar{v}_n+c_{n+1}\bar{v}_{n+1}=\bar{0}}$, with $\mathbf{c_1 \neq 0}$.

So W **is a set of linearly dependent vectors.**

5. Let $\bar{w}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \bar{w}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad \bar{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Is $\bar{p}$ in Span$(\bar{w}_1, \bar{w}_2)$. If so, express $\bar{p}$ as a linear combination of $\bar{w}_1, \bar{w}_2$

$$\left[\begin{array}{rr|r} 2 & 3 & 1 \\ -3 & 6 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rr|r} -1 & 9 & 2 \\ -3 & 6 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rr|r} 1 & -9 & -2 \\ -3 & 6 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rr|r} 1 & -9 & -2 \\ 0 & -21 & -5 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{rr|r} 1 & -9 & -2 \\ 0 & 1 & 5/21 \end{array}\right] \Rightarrow \left[\begin{array}{rr|r} 1 & 0 & 3/21 \\ 0 & 1 & 5/21 \end{array}\right]$$

So $\bar{p}$ **IS in Span**$(\bar{w}_1, \bar{w}_2)$: $\quad \mathbf{\bar{p} = \frac{3}{21}\bar{w}_1 + \frac{5}{21}\bar{w}_2}$

Consider the set of vectors, $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 9 \\ 10 \end{bmatrix} \right\}$

Is this set linearly dependent or linearly independent. Cite and use a specific theorem to prove your answer.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 8 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 8 & 9 & 10 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5/4 & 6/4 \\ 0 & 0 & 0 & 8 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5/4 & 6/4 \\ 0 & 0 & 0 & 0 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5/4 & 6/4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Since the number of non-zero rows in Reduced Row Echelon form (the Rank) is equal to the number of original vectors, (so no rows have been "zero'd out"), our vectors are **linearly independent**.

6. Give an example of each of the following:

(a) A set of two vectors in R^4 which are linearly dependent.

TWO vectors are linearly dependent if and only if one is a scalar multiple of the other:

Example: $[\mathbf{3}, \mathbf{1}, \mathbf{-4}, \mathbf{1}]$ and $[\mathbf{9}, \mathbf{3}, \mathbf{-12}, \mathbf{3}]$

(b) An orthogonal set of three vectors in R^3.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) Two vectors for which the cross product does not exist.

Only vectors in R^3 can be operands of a cross product...

Example: $[\mathbf{3}, \mathbf{1}, \mathbf{-4}, \mathbf{1}]$ and $[\mathbf{9}, \mathbf{3}, \mathbf{-12}, \mathbf{3}]$

(d) Three non-zero vectors, $\bar{u}, \bar{v}, \bar{w}$, for which $\text{Span}(\{\bar{u}, \bar{v}, \bar{w}\}) = \text{Span}(\{\bar{u}, \bar{v}\})$.

$$\bar{\mathbf{u}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \bar{\mathbf{w}} = \begin{bmatrix} 3 \\ 3 \\ 24 \end{bmatrix}$$