Math 214 - Fall 2019 - Final Exam

1. For each statement below, CLEARLY write **T** for True, **F** for False:

   (a) **T** For a matrix, $A_{m\times n}$, where $n \neq m$, $Dim(Col(A)) = Dim(Row(A))$

   (b) **T** The orthogonal complement of the subspace Span$\left(\{\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}\}\right)$ is Span$\left(\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\}\right)$

   (c) **T** Let vectors $\overline{u}, \overline{v} \in R^3$, with neither vector equal to $\overline{0}$. Further, $\overline{u} \cdot \overline{v} = 0$. Then the three vectors $\overline{u}, \overline{v}, \overline{u} \times \overline{v}$ form an orthogonal set of vectors.

   (d) **F** There exists an orthogonal matrix, U, for which $U\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

   (e) **T** There exists an orthogonal matrix, U, for which $U\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

   (f) **F** There exists some "two by three" matrix A for which: $Row(A) = Col(A) = Null(A)$

   (g) **T** Let $\overline{v}$ be an eigenvector for A. Then $\bar{v}$ is an eigenvector for A^{-1}.

   (h) **T** Let $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 5x - y \\ y + 2z \end{bmatrix}$. T is a linear transformation from R^3 to R^2.

   (i) **F** The Gorshgorin Theorem is a highly accurate method for approximating the eigenvalues of a matrix.

   (j) **T** The rows of a matrix with eigenvalues: $\lambda = 1.5, 8.2, 27, 0, -5.2, 1, \frac{1}{2}$ form a set of seven linearly dependent vectors.

2. (a) Determine all solutions of the system of equations below. Express your solution in vector form.

$$w + 2x - 5y = -8 \qquad 2x + 6y - 4z = -14 \qquad 2w + 2x - 16y + 4z = -2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -5 & 0 & -8 \\ 0 & 2 & 6 & -4 & -14 \\ 2 & 2 & -16 & 4 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -5 & 0 & -8 \\ 0 & 2 & 6 & -4 & -14 \\ 0 & -2 & -6 & 4 & 14 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -5 & 0 & -8 \\ 0 & 1 & 3 & -2 & -7 \\ 0 & -2 & -6 & 4 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -5 & 0 & -8 \\ 0 & 1 & 3 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -11 & 4 & 6 \\ 0 & 1 & 3 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11y - 4z + 6 \\ -3y + 2z - 7 \\ y \\ z \end{bmatrix} = \mathbf{y} \begin{bmatrix} 11 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \mathbf{z} \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

(b) State the Rank Theorem, and verify that it holds for the corresponding homogenous system of equations.

The corresponding homogenous system, row reduced augmented matrix, and solution are:

$$w + 2x - 5y = 0 \qquad 2x + 6y - 4z = 0 \qquad 2w + 2x - 16y + 4z = 0$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -11 & 4 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11y - 4z \\ -3y + 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 11 \\ -3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Rank Theorem: For an m by n matrix, A:

$$\mathbf{Rank(A) + Nullity(A) = n}$$

For the corresponding homogenous system is:

$$\mathbf{Rank(A) + Nullity(A) = 2 + 2 = 4 = n}$$

3. Let $\overline{p} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\overline{q} = \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$, $\overline{r} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix}$, $\overline{s} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$.

Determine the following quantities or state that it is not possible.

(a) $\overline{p} \times \overline{q}$

$$\overline{p}\times\overline{q} = Det\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ -5 & 2 & 4 \end{vmatrix} = [(-3)\cdot 4 - 1\cdot 2,\ \ 1\cdot(-5) - 2\cdot 4,\ \ 2\cdot 2 - (-3)\cdot(-5)]$$

$$= [\mathbf{-14}, \mathbf{-13}, \mathbf{-11}]$$

(b) $(\overline{r}\cdot\overline{s})\overline{q}$

$$(\overline{r}\cdot\overline{s})\overline{q} = (0\cdot 3 + 1\cdot 1 + (-3)\cdot 4 + 2\cdot 1)\begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix} = \mathbf{-9}\begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$$

(c) $\overline{r} \times \overline{s}$

Impossible: Cross products only apply to vectors in R^3

(d) $Proj_{\overline{s}}(\overline{r})$

$$Proj_{\overline{s}}(\overline{r}) = \frac{\overline{r}\cdot\overline{s}}{\overline{s}\cdot\overline{s}}\ \overline{s} = -\frac{\mathbf{9}}{\mathbf{27}}\begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix}$$

(e) $\|\overline{r} - \overline{s}\|$

$$\|\overline{r} - \overline{s}\| = \sqrt{(0-3)^2 + (1-1)^2 + (-3-4)^2 + (2-1)^2} = \sqrt{\mathbf{149}}$$

4. Determine the plane in R^3 which contains the points $P = (2, 1, 3)$, $Q = (3, 2, 3)$, $R = (7, 1, 4)$. Express your answer in both i) general form and ii) vector form.

Define $\overline{v} = Q - p = [1, 1, 0]$, $\quad \overline{w} = R - P = [5, 0, 1]$

$\overline{v}$ and $\overline{w}$ are both in the plane, therefore the normal to this plane is:

$$\overline{n} = \overline{v} \times \overline{w} = Det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 5 & 0 & 1 \end{vmatrix} = [\mathbf{1}, \mathbf{-1}, \mathbf{-5}]$$

Our plane is the set of all $\overline{x} = [x, y, z]$ for which:

Vector form: $\overline{n} \cdot (\overline{x} - P) = [\mathbf{1}, \mathbf{-1}, \mathbf{-5}] \cdot [\mathbf{x - 2}, \mathbf{y - 1}, \mathbf{z - 3}] = \mathbf{0}$

General form: $x - 2 - (y - 1) - 5(z - 3) = \mathbf{x - y - 5z + 14 = 0}$

Determine the line in R^7 which contains the points $P = (8, 6, 7, 5, 3, 0, 9)$ and $Q = (2, 5, 9, 1, 3, 0, 3)$ in vector form.

$$\text{Define } \overline{v} = P - Q = \begin{bmatrix} 6 \\ 1 \\ -2 \\ 4 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

$$\text{Then our line is given by: } \overline{x(t)} = P + t\overline{v} = \begin{bmatrix} 8 \\ 6 \\ 7 \\ 5 \\ 3 \\ 0 \\ 9 \end{bmatrix} + \mathbf{t} \begin{bmatrix} 6 \\ 1 \\ -2 \\ 4 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

5. Prove the following statements using facts that we have emphasized during the course:

   (a) Let the square matrices, A and B be similar, with similarity transform $P\,B\,P^{-1} = A$. Then the traces of A and B are equal: $Tr(A) = Tr(B)$

   Let A and B both be n by n matrices. Assume that the eigenvalues of A are $\lambda_1, \lambda_2, ...\lambda_n$, and that the eigenvalues of B are $\mu_1, \mu_2, ...\mu_n$.

   Since A and B are similar ($A \sim B$), WLOG: $\lambda_k = \mu_k, k = 1, 2, ...n$.

   Finally, since we have proven that the trace of a square matrix is equal to the sum of the eigenvalues:

   Then: $\mathbf{Trace(A)} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \lambda_{\mathbf{k}} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \mu_{\mathbf{k}} = \mathbf{Trace(B)}$

   (b) Let $V = \{\overline{v}_1, \overline{v}_2, ...\overline{v}_k\}$ by an orthogonal set of k vectors in R^n. Then V is a linearly independent set as well.

   Construct our test equation: $c_1\overline{v}_1 + c_2\overline{v}_2 + c_3\overline{v}_3 + ... + c_n\overline{v}_n = \overline{0}$

   For each vector, $\overline{v}_k, k = 1...n$, take the dot product of both sides of the test equation with $\overline{v}_k$:

   $$\overline{v}_k(c_1\overline{v}_1 + c_2\overline{v}_2 + c_3\overline{v}_3 + ... + c_n\overline{v}_n) = \overline{v}_k \cdot \overline{0} = 0$$

   $$c_1\overline{v}_k \cdot \overline{v}_1 + c_2\overline{v}_k \cdot \overline{v}_2 + c_3\overline{v}_k \cdot \overline{v}_3 + ... + c_n\overline{v}_k \cdot \overline{v}_n = 0$$

   Further, since no vector in V can be the $\overline{0}$ vector, $\overline{\mathbf{v}}_\mathbf{i} \cdot \overline{\mathbf{v}}_\mathbf{i} > \mathbf{0}$

   Since V is an orthogonal set, $\overline{\mathbf{v}}_\mathbf{i} \cdot \overline{\mathbf{v}}_\mathbf{j} = \mathbf{0}$ for $\mathbf{i} \neq \mathbf{j}$.

   So all terms become 0 except for $\overline{v}_k \cdot \overline{v}_k = 0$. This forces:

   $$c_k\overline{v}_k \cdot \overline{v}_k = 0.$$

   Therefore, $\mathbf{c_k = 0}$. Since this is true for all k from 1 to n, we have proven that **V is a linearly independent set of vectors.**

6. (a) Compute the determinant of $A = \begin{bmatrix} 0 & \mathbf{-2} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & 0 & 3 & 5 \\ 0 & \mathbf{1} & \mathbf{0} & \mathbf{3} \\ 0 & \mathbf{0} & \mathbf{5} & \mathbf{0} \end{bmatrix}$

$$Det(A) = (-1)^{(2+1)} \cdot 2 \cdot Det \begin{vmatrix} \mathbf{-2} & 1 & \mathbf{2} \\ \mathbf{1} & 0 & \mathbf{3} \\ 0 & \mathbf{5} & 0 \end{vmatrix}$$

$$= (-1)^{(1+2)} \cdot 2 \cdot (-1)^{(3+2)} \cdot 5 \cdot Det \begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= (-1)^{(1+2)} \cdot 2 \cdot (-1)^{(3+2)} \cdot 5 \cdot (-8) = \mathbf{-80}$$

(b) Is $\overline{p} = \begin{bmatrix} \pi \\ e \\ -1 \\ \frac{1}{19} \end{bmatrix}$ an element in the subspace $S = Span\left(\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 0 \end{bmatrix} \right\}\right)$? Explain.

YES! Since the matrix A above has a non-zero determinant, the four columns of A are a linearly independent set of vectors in R^4, spanning R^4. Since they span R^4, any vector in R^4 is a linear combination of these four vectors. Of course, this includes $\overline{p}$.

7. Eigen-stuff!

(a) $A = \begin{bmatrix} 4 & 8 & -3 \\ 0 & -2 & 0 \\ -3 & 0 & 4 \end{bmatrix}$. One of the eigenvectors of A is: $\bar{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Find all eigenvalues and eigenvectors.

i. Let $\mathbf{\bar{v}_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Multiplying, we see that: $A\bar{v}_1 = 7\bar{v}_1$. So, $\boldsymbol{\lambda_1 = 7}$

ii. Compute $A - \lambda I = \begin{bmatrix} 4-\lambda & 8 & -3 \\ 0 & -2-\lambda & 0 \\ -3 & 0 & 4-\lambda \end{bmatrix}$.

Clearly, $Det(A - \lambda I)$ has a factor: $-2 - \lambda$, so $\boldsymbol{\lambda_2 = -2}$.

iii. Since $Trace(A) = 6 = \lambda_1 + \lambda_2 + \lambda_3 = 7 + -2 + \lambda_3$, $\boldsymbol{\lambda_3 = 1}$

iv. Solving $(A+2I)\bar{v}_2 = \bar{0}$ and $(A-I)\bar{v}_3 = \bar{0}$, we find our last two eigenvectors:

$$\mathbf{\bar{v}_2} = \begin{bmatrix} 16 \\ -9 \\ 8 \end{bmatrix} \qquad \mathbf{\bar{v}_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(b) Verify that any matrix of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, has the following eigenvalue / eigenvector pairs:

$$\lambda_1 = a, \bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = c, \bar{v}_2 = \begin{bmatrix} b \\ c-a \end{bmatrix}$$

Verifying the relation: $A\bar{v} = \lambda\bar{v}$, we compute:

$$A\bar{v}_1 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{\lambda_1 = a} \ \checkmark$$

$$A\bar{v}_2 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} b \\ c-a \end{bmatrix} = \begin{bmatrix} ab+bc-ba \\ c(c-a) \end{bmatrix} = c\begin{bmatrix} b \\ c-a \end{bmatrix} \Rightarrow \boldsymbol{\lambda_2 = c} \ \checkmark$$

(c) Use the pervious part to compute B^{2019} for $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$

Since matrix B is or the form from the previous part, we identify:

$\lambda_1 = -1, \overline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\qquad \lambda_1 = -1, \overline{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \overline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then: $S = [\overline{v}_1 \mid \overline{v}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\qquad S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Finally: $\mathbf{B^{2019}} = SD^{2019}S^{-1} = S \begin{bmatrix} -1^{2019} & 0 \\ 0 & 1^{2019} \end{bmatrix} S^{-1}$

$$= S \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} = SDS^{-1} = \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

(d) Recalling from Calculus I that:

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - ...,$$

compute the matrix $\cos(B)$.

We proceed in the following manner:

i. From above: $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\qquad S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

ii. $D^2 = I$, $D^{2k} = I$ for $k = 0, 1, 2, 3, ...$

iii. $\cos(D) = I - \frac{1}{2}D^2 + \frac{1}{24}D^4 - \frac{1}{720}D^6 + \frac{1}{40320}D^8 - ...$

$$= \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}D^{2k} = \left(\sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}\right) I = \cos(1) \cdot I = \begin{bmatrix} \cos(1) & 0 \\ 0 & \cos(1) \end{bmatrix}$$

iv. $\cos(\mathbf{B}) = \mathbf{S}\ \cos(\mathbf{D})\ \mathbf{S^{-1}} = \mathbf{S}\ \cos(\mathbf{1})\ \mathbf{I}\ \mathbf{S^{-1}} = \cos(\mathbf{1})\ \mathbf{S}\ \mathbf{I}\ \mathbf{S^{-1}} = \cos(\mathbf{1})\ \mathbf{I}$

8. Subspaces

From a previous problem, we might have seen a matrix similar to: $A = \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 2 & 6 & -4 \\ 2 & 2 & -16 & 4 \end{bmatrix}$.

Determine:

(a) The Row Space of A, $Row(A) =$

From that same previous problem, we have already computed the reduced row echelon form:

$$RREF(A) = \begin{bmatrix} 1 & 0 & -11 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So: $\mathbf{Row(A) = Span(\{[1, 0, -11, 4], [0, 1, 3, -2]\})}$

(b) The Column Space of A, $Col(A) =$

Since columns $1, 2$ of $RREF(A)$ span the column space of $RREF(A)$, the same columns of A span $Col(A)$:

$$\mathbf{Col(A) = Span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right\}\right)$$

(c) The Null Space of A, $Nul(A) =$

The space $Null(A)$ is the same space as the space of homogenous solutions from that same previous problem:

$$\mathbf{Null(A) = Span}\left(\left\{\begin{bmatrix} 11 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

(d) Compute $Rank(A)$ and $Nullity(A)$

$\mathbf{Rank(A) = Dim(Col(A)) = Dim(Row(A)) = 2}$

$\mathbf{Nullity(A) = Dim(Null(A)) = 2}$

9. Consider the subspace of R^4, $S = Span\left(\left\{\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\0\end{bmatrix}\right\}\right)$.

Determine the Orthogonal Complement of S:

Let $\bar{w} = \begin{bmatrix}p\\q\\r\\s\end{bmatrix}$ be a member of the Orthogonal Complement of S.

Then the dot product of $\bar{w}$ with either vector above is equal to 0. This produces two equations:

$$\mathbf{p - r + 2s = 0}, \text{ and } \mathbf{p + 2q + 3r = 0}.$$

Next, we solve by row reducing:

$$\left[\begin{array}{cccc|c}1 & 0 & -1 & 2 & 0\\1 & 2 & 3 & 0 & 0\end{array}\right] \Rightarrow \left[\begin{array}{cccc|c}1 & 0 & -1 & 2 & 0\\0 & 2 & 4 & -2 & 0\end{array}\right] \Rightarrow \left[\begin{array}{cccc|c}1 & 0 & -1 & 2 & 0\\0 & 1 & 2 & -1 & 0\end{array}\right]$$

The solutions to this system are: $\begin{bmatrix}p\\q\\r\\s\end{bmatrix} = \begin{bmatrix}r-2s\\-2r+s\\r\\s\end{bmatrix} = \mathbf{r}\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \mathbf{s}\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$

So the Orthogonal Complement is: **Span** $\left(\left\{\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}\right\}\right)$.

Prove that for any matrix A, the subspace $Row(A)$ is the orthogonal complement of $Null(A)$. Hint: start with "let $\bar{v}$ be any vector in $Row(A)$, and let $\overline{w}$ be any vector in $Null(A)$. Then..."

Let $\bar{v} \in Row(A), \bar{w} \in Null(A)$. Then $\mathbf{A\bar{w} = \bar{0}}$. Therefore, for the $k-th$ row of A, $\mathbf{A_{row\ k} \cdot \bar{w} = 0}$. Since any $\bar{v} \in Row(A)$ is a linear combination of the rows of A, we have:

$$\mathbf{\bar{v} \cdot \bar{w} = (\sum_{k=1}^{n} c_k A_{row\ k}) \cdot \bar{w} = \sum_{k=1}^{n} c_k (A_{row\ k} \cdot \bar{w}) = \sum_{k=1}^{n} c_k \cdot 0 = 0}.$$

Since, for any $\bar{v} \in Row(A), \bar{w} \in Null(A)$, $\bar{v} \cdot \bar{w} = 0$, $Row(A)$ and $Null(A)$ are orthogonally complement!

10. Inverses: $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 3 \end{bmatrix}$

(a) Compute the matrix A^{-1}:

Performing row operations on $[A|I]$ to produce $[I|A^{-1}]$:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 4 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 4 & 0 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 0 & 3 \end{array}\right]$$

The inverse, then, is: $\mathbf{A^{-1}} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -4 & 0 & 3 \end{bmatrix}$

(b) Prove: For an invertible n by n matrix, B, if v is an eigenvector of B, then v is also an eigenvector of B^{-1}.

If v is an eigenvector of B, then $\mathbf{Bv} = \lambda\mathbf{v}$, for some λ. Since B is invertible, $\lambda \neq 0$.

Multiplying on the left by B^{-1}, we have: $\mathbf{B^{-1}Bv} = \mathbf{B^{-1}}\lambda\mathbf{v}$.

Simplifying: $v = \lambda B^{-1}v$.

Dividing both sides by λ, we have: $\frac{\mathbf{1}}{\lambda}\mathbf{v} = \mathbf{B^{-1}v}$.

Therefore, $\mathbf{v}$ is in fact an eigenvector of B^{-1} corresponding to eigenvalue $\frac{\mathbf{1}}{\lambda}$.

11. Consider the two vectors $\overline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \overline{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(a) Find a third vector, $\overline{v}_3$, which is orthogonal to both $\overline{v}_1$ and $\overline{v}_2$. Hint: the trick is found somewhere else in this exam.

Since neither $\overline{v}_1$ or $\overline{v}_2$ are $\overline{0}$, nor are they parallel, we can compute a non-zero cross product, $\overline{v}_3$ which is orthogonal to both $\overline{v}_1$ and $\overline{v}_2$:

$$\overline{v}_3 = \overline{v}_1 \times \overline{v}_2 = Det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 2 & -2 & 1 \end{vmatrix} = [\mathbf{-4}, \mathbf{-5}, \mathbf{-2}]$$

(b) Use these three vectors to construct an orthogonal matrix, U.

We also notice that $\overline{v}_1 \cdot \overline{v}_2 = 0$. So the three vectors, $V = \{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$ form an orthogonal set. But in order for these to form the columns of an orthogonal matrix, they must first be **NORMALIZED**!

We compute: $\|\overline{v}_1\| = \sqrt{5}$, $\|\overline{v}_2\| = 3$, $\|\overline{v}_3\| = 3\sqrt{5}$

So our orthogonal matrix is: $U = \left[\frac{\overline{v}_1}{\|\overline{v}_1\|} \Big| \frac{\overline{v}_2}{\|\overline{v}_2\|} \Big| \frac{\overline{v}_3}{\|\overline{v}_3\|} \right] = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2}{3} & -\frac{4\sqrt{5}}{15} \\ 0 & -\frac{2}{3} & -\frac{5\sqrt{5}}{15} \\ -\frac{2\sqrt{5}}{5} & \frac{1}{3} & -\frac{2\sqrt{5}}{15} \end{bmatrix}$

(c) Compute the inverse, U^{-1}.

Since U is an orthogonal matrix, $U^{-1} = U^T = \begin{bmatrix} \frac{\sqrt{5}}{5} & 0 & -\frac{2\sqrt{5}}{5} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{4\sqrt{5}}{15} & -\frac{5\sqrt{5}}{15} & -\frac{2\sqrt{5}}{15} \end{bmatrix}$

12. Write down as many equivalent facts from the Fundamental Theorem of Invertible Matrices as you can, (as you did in Quiz 4), numbering them. Then: Choose any two of these facts, and prove that one of them implies the other.

From the totality of chapters we've read in the text, we have encountered "The Fundamental Theorem of Linear Algebra". Over the last few chapters (*), it has grown to include many statements which are all equivalent. Indeed, this theorem starts with the claim: **The following are equivalent:**, leading some to refer to this as a TFAE theorem! Below, list as many of these equivalent statements as possible.

(a) A is invertible.

(b) $Ax = b$ has a unique solution for every b in R^n.

(c) $Ax = 0$ has only the trivial solution.

(d) The reduced row echelon form of A, $RREF(A)$, is equal to I_n.

(e) A is the product of elementary matrices.

(f) $\text{rank}(A) = n$.

(g) $\text{nullity}(A) = 0$.

(h) The column vectors of A are linearly independent.

(i) The column vectors of A form a basis for R^n.

(j) The column vectors of A span R^n.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A form a basis for R^n.

(m) The row vectors of A span R^n.

(n) $\det(A) \neq 0$.

(o) 0 is not an eigenvalue of A.

Since there are 210 possible variations on "prove that part i implies part j", we leave this to your text ... or your memory.

(*) Linear Algebra, a Modern Introduction, 4th edition.
Author: David Poole, 2015, Cengage Learning. ISBN-13: 978-1-285-46324-7.